Math 4200
Wednesday August 26

1.1-1.2 Algebra and geometry of complex arithmetic, continued.
We'll pick up in Monday's notes where we left off (there was still a lot to talk about there), and used today's notes to talk about solutions to polynomial equations.

Announcements: We'll try group quizzes at the end of class today!

Warm-up example: Use rectangular coordinates to find all complex solutions to the following. Sketch the solutions in the complex plane.

a) $z^2 = -9$

b) $z^3 = 1$.

After we discuss the polar form of complex numbers, we'll come back re-solve the page 1 equations that way.

a) $z^2 = -9$

b) $z^3 = 1$.

Solving polynomial equations in $\mathbb{C}$ (section 1.1)

1) Every non-zero complex number z_0 has two square roots, i.e solutions z to

$$z^2 = z_0$$

and they are opposites.

<u>proof 1</u>: Use the polar form, writing $z_0 = \rho\, e^{i\phi}$, $z = r\, e^{i\,\theta}$, with $r, \rho > 0$.

<u>proof 2</u>: (To convince you how great polar form is) Use rectangular coordinates: Express z_0, z in terms of their real and imaginary parts,

$$z_0 = x_0 + i\, y_0$$
$$z = x + i\, y$$
$$(x + i\, y)^2 = x_0 + i\, y_0$$
$$\begin{cases} x^2 - y^2 = x_0 \\ 2\, x\, y = y_0 \end{cases}$$

Case 1: If $y_0 \neq 0$ then $x, y \neq 0$. Solve for y from the second equation and subsitute into the first:

$$x^2 - \left(\frac{y_0}{2\, x}\right)^2 = x_0$$
$$4\, x^4 - 4\, x_0 x^2 - y_0^2 = 0.$$

Use the quadratic formula for real coefficients for x^2

$$x^2 = \frac{4\, x_0 + \sqrt{16\, x_0^2 + 16\, y_0^2}}{8} = \frac{x_0 + \sqrt{x_0^2 + y_0^2}}{2}.$$

There are two opposite real values of x which solve this equation, with corresponding opposite values of $y = \dfrac{y_0}{2\, x}$.

Case 2: If $y_0 = 0$ it meant that $z_0 = x_0$ was real, and you already know how to find the two square roots. If $x_0 > 0$ they will be real square roots, and if $x_0 < 0$ they will be imaginary.

2) The general degree n polynomial equation

$$p(z) := z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0.$$

$$a_{n-1}, \ldots a_1, a_0 \in \mathbb{C}$$

You've been told forever that every degree n polynomial equation has n complex roots, counting multiplicty. This fact is known as "The Fundamental Theorem of Algebra." You'll learn a beautiful "elementary" proof of the fundamental theorem of algebra in this class. It's a proof by contradiction though, and except for very special polynomials there are no explicit formulas for exact solutions.....

There is a very complictaed cubic formula for cubic equations. There is also a formula for the roots of 4^{th} order polynomials. The Abel-Ruffini Theorem asserts however, that there are no general formulas for the roots of degree 5 and higher polynomial equations, such that these formulas use only the algebraic operations of addition, multiplication, and taking radicals (square roots, cube roots, etc.). One of the founders of number theory, Évariste Galois, developed "Galois Theory", which explains exactly which special higher degree polynomial equations can be solved using these operations. These are topics in advanced algebra courses.

3) The special polynomial equation $z^n = 1$.
Its solutions are called "the n^{th} roots of unity", and there are n of them.

Since complex multiplication in polar form reads

$$z\,w = r\,e^{i\,\theta}\,\rho\,e^{i\,\phi} = r\,\rho\,e^{i\cdot(\theta+\phi)},$$

(where $r = |z|$, $\theta = arg(z)$, $\rho = |w|$, $\phi = arg(w)$), it's easy to check via induction, that

$$z^n = r^n\,e^{i\,n\,\theta}\,.$$

This formula for powers is known as "DeMoivre's formula".

So, to solve $z^n = 1$, express $z = |z|\,e^{i\,\theta}$ and solve

$$|z|^n\,e^{i\,n\,\theta} = 1\,.$$